

Bounding Risk Measures for Portfolios with Known Marginal Risks

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The problem at hand

Consider a function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ and a random vector

$$X := (X_1, \dots, X_n)$$

of n one-period financial losses or insurance claims
on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

The Value-at-Risk (quantile) at level α for the aggregate loss $\psi(X)$ can be computed once we know the joint distribution of the vector X , i.e.

$$F(x_1, \dots, x_n) = \mathbb{P}[X_1 \leq x_1, \dots, X_n \leq x_n].$$

Unfortunately, the distribution function (df) of the random vector X is **not** completely determined by the F_i 's.

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Unfortunately, the distribution function (df) of the random vector X is **not** completely determined by the F_i 's.

There are infinitely many distributions for the vector X which are consistent with the initial choice of the marginals.

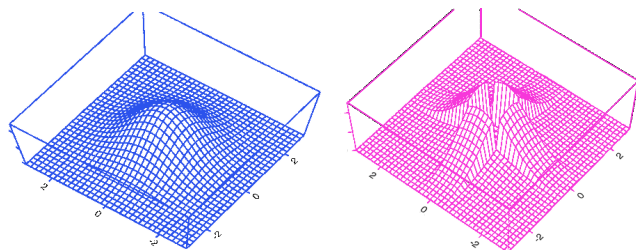


Figure: Two different bivariate dfs having $N(0, 1)$ -marginals and the same correlation

Which is the df giving the worst-possible Value-at-Risk (VaR) for the random variable $\psi(X)$?

History of the problem

- Makarov (1981) provided the first result for $n = 2$; $\psi = +$.
- Frank et al. (1987) restated Makarov's result, using copulas.
- Independently, Rüschendorf (1982) gave a more elegant proof of the same theorem using duality.
- Williamson and Downs (1990) introduced the use of dependence information.
- Embrechts et al. (2003) found the solution in the presence of partial information for non-decreasing functions ψ .
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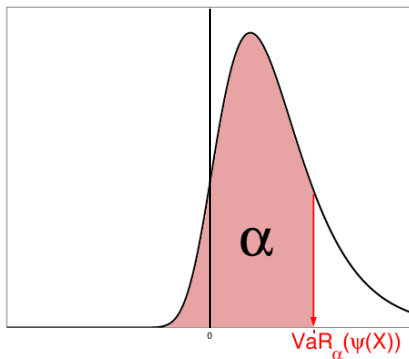
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Value-at-Risk for the aggregate loss

Definition

For $\alpha \in [0, 1]$, the *Value-at-Risk* at probability level α for $\psi(X)$ is its α -quantile, defined as $\text{VaR}_\alpha(Y) := G^{-1}(\alpha)$, where G is the df of $\psi(X)$.



Searching for the worst-possible VaR means looking for

$$m_\psi(s) := \inf\{\mathbb{P}[\psi(X) < s] : X_i \sim F_i, i = 1, \dots, n\}.$$

Indeed, according to the definition of VaR, we have

$$\text{VaR}_\alpha(\psi(X)) \leq m_\psi^{-1}(\alpha), \alpha \in [0, 1].$$

The distribution of $\psi(X)$ can be uniquely defined through the marginal dfs and their interdependence, which can be modeled by the concept of **copula**.

Definition

A *copula* is any n -dimensional df restricted to $[0, 1]^n$ having standard uniform marginals.

Given a copula C and a set of n marginals F_1, \dots, F_n one can always define a df F on \mathbb{R}^n having these marginals by

$$F(x_1, \dots, x_n) := C(F_1(x_1), \dots, F_n(x_n)). \quad (1)$$

Sklar's theorem states conversely that we can always find a copula C coupling the marginals of a fixed df F through (1).

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Sklar's theorem states conversely that we can always find a copula C coupling the marginals of a fixed df F through (1).

- **independent** marginals are merged by the

$$\Pi : [0, 1]^n \rightarrow [0, 1]; \Pi(u_1, \dots, u_n) := \prod_{i=1}^n u_i$$

- **comonotonic** marginals are merged by the so-called *upper Fréchet bound*

$$M : [0, 1]^n \rightarrow [0, 1]; M(u_1, \dots, u_n) := \min\{u_1, \dots, u_n\}$$

- **countermonotonic** marginals are merged by the so-called *lower Fréchet bound*

$$W : [0, 1]^n \rightarrow [0, 1]; W(u_1, \dots, u_n) := \left[\sum_{i=1}^n u_i - n + 1 \right]^+$$

Any copula C lies between the lower and upper Fréchet bounds:

$$W \leq C \leq M.$$

Dependence information

By Sklar's theorem, our problem can be equivalently expressed as

$$m_\psi(s) = \inf \{ \mathbb{P}_C [\psi(X) < s] : C \in \mathfrak{C}_n \},$$

where \mathfrak{C}_n denotes the set of all n -dimensional copulas.

Putting a lower bound on the copula C of the portfolio can be interpreted as having partial information regarding the dependence structure of our portfolio of risks.

If this is the case, the problem reduces to

$$m_{C_L, \psi}(s) := \inf \{ \mathbb{P}_C [\psi(X) < s] : C \geq C_L \}.$$

If $C_L = W$, then come back to our original problem.

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Main Result with Dependence information

**When a lower copula-bound on the portfolio copula C is assumed,
the problem at hand is fully solved.**

Define

$$\tau_{C,\psi}(F_1, \dots, F_n)(s) := \sup_{x_1, \dots, x_{n-1} \in \mathbb{R}} C(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n^-(\widehat{\psi}_{x_{-n}}(s))).$$

Theorem 1 Let $X = (X_1, \dots, X_n)$ be a random vector on \mathbb{R}^n ($n > 1$) having marginal dfs F_1, \dots, F_n and copula C . Assume that there exists a copula C_L such that $C \geq C_L$. If $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is non-decreasing in each coordinate, then, for every $\alpha \in [0, 1]$, we have

$$\text{VaR}_\alpha(\psi(X)) \leq \tau_{C_L, \psi}(F_1, \dots, F_n)^{-1}(\alpha). \quad (2)$$

Theorem 2 Assume ψ is also right-continuous in its last argument and define the function $C_t : [0, 1]^n \rightarrow [0, 1]$ as follows:

$$C_t(u) := \begin{cases} \max\{t, C_L(u)\} & \text{if } u = (u_1, \dots, u_n) \in [t, 1]^n, \\ \min\{u_1, \dots, u_n\} & \text{otherwise,} \end{cases}$$

where $t = \tau_{C_L, \psi}(F_1, \dots, F_n)^{-1}(\alpha)$. Then C_t is a copula and it attains bound (2), i.e. under C_t we have

$$\text{VaR}_\alpha(\psi(X)) = t.$$

Important Remark on the Theorem

- A priori assumptions such as $C \geq \Pi$ may lead to a critical undervaluation of the portfolio risk since the componentwise ordering in the class \mathfrak{C}^n is not complete.

Main Result without information on dependence

Consider now

$$C_L = W.$$

The fact that the function W is not a copula for $n \geq 3$ causes problems.

Though the *standard* bound stated in Theorem 1 holds in arbitrary dimension, when

$$n > 2 \text{ and } C_L = W,$$

it may fail to be sharp.

Mutually Exclusive Risks

Consider a portfolio of *mutually exclusive risks*, i.e. risks that can be positive at most one at a time.

In this specific case, of actuarial interest, we have that
the lower Fréchet bound W is a proper df
 and the standard bound is then sharp for arbitrary n ,
 even in the no-information scenario.

In the no-information scenario, it is convenient to express our problem using a duality result given in Rüschendorf (1982):

$$\begin{aligned}
 m_\psi(s) &= \inf\{\mathbb{P}[\psi(X) < s] : X_i \sim F_i, i = 1, \dots, n\} \\
 &= 1 - \inf \left\{ \sum_{i=1}^n \int f_i dF_i : f_i \in L^1(F_i), i \in N \text{ s.t.} \right. \\
 &\quad \left. \sum_{i=1}^n f_i(x_i) \geq 1_{[s, +\infty)}(\psi(x)) \text{ for all } x \in \mathbb{R}^n \right\}.
 \end{aligned}$$

Some remarks on the dual problem

- The dual optimization problem seems to be very difficult to solve;
- Explicit results are known only for uniformly or binomially distributed risks;
- Unfortunately, the solution in the case of the sum of uniform marginals does not work in the general case.

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Dual bounds

We use the dual problem to provide a bound which is better (i.e. \geq) than the *standard* one.

Theorem 3 Let F be a non-negative, continuous df. If $F_i = F, i = 1, \dots, n$, then for every $s \geq 0$,

$$m_+(s) \geq 1 - n \inf_{r \in [0, s/n]} \frac{\int_r^{s-(n-1)r} (1 - F(x)) dx}{s - nr}.$$

- For $n = 2$ this theorem gives the sharp bound already stated
- This *dual* bound is strictly larger than the standard bound for most dfs and thresholds s of interest
- For $n > 2$, this bound can be easily calculated numerically.

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- For $n > 2$, this bound can be easily calculated numerically.

The assumptions under which Theorem 3 is valid are considerable with respect to the setting of the previous sections,

but still consistent with most dfs F and thresholds s of actuarial/financial interest.

Under such assumptions, it is easy to show that, for s large enough, the standard bound reduces to

$$\tau_{W,+}(F, \dots, F)(s) = [nF(s/n) - n + 1]^+.$$

How can we compare the quality of the dual bound with respect to the standard bound generally used in the literature?

We define the two dfs $\underline{F}_N, \bar{F}_N$ by

$$\underline{F}_N(x) := \frac{1}{N} \sum_{i=1}^N 1_{[q_r, +\infty)}(x),$$

$$\bar{F}_N(x) := \frac{1}{N} \sum_{i=0}^{N-1} 1_{[q_r, +\infty)}(x),$$

the jump points q_0, \dots, q_N being the quantiles of F defined by

$$q_0 := \inf \text{supp}(F), q_N := \sup \text{supp}(F) \text{ and}$$

$$q_r := F^{-1}(r/N), r = 1, \dots, N-1.$$

It is straightforward that

$$\underline{F}_N \leq F \leq \overline{F}_N,$$

from which it follows that

$$\underline{m}_+(s) \leq m_+(s) \leq \overline{m}_+(s),$$

where $\underline{m}_+(s)$ and $\overline{m}_+(s)$ are naturally defined as:

$$\underline{m}_+(s) := \inf \left\{ \mathbb{P} \left[\sum_{i=1}^n X_i < t \right] : X_i \sim \underline{F}_N, i = 1, \dots, n \right\},$$

$$\overline{m}_+(s) := \inf \left\{ \mathbb{P} \left[\sum_{i=1}^n X_i < t \right] : X_i \sim \overline{F}_N, i = 1, \dots, n \right\}.$$

Given that \underline{F}_N is a (possibly defective) discrete df, $\underline{m}_+(s)$ is the solution of the following LP:

$$\underline{m}_+(s) = \min_{p_{j_1, \dots, j_n}} \sum_{j_1=1}^N \cdots \sum_{j_n=1}^N p_{j_1, j_2, \dots, j_n} 1_{(-\infty, t)} \left(\sum_{i=1}^n q_{j_i} \right) \text{ subject to}$$

$$\left\{ \begin{array}{ll} \sum_{j_2=1}^N \sum_{j_3=1}^N \cdots \sum_{j_n=1}^N p_{j_1, \dots, j_n} = \frac{1}{N} & j_1 = 1, \dots, N, \\ \sum_{j_1=1}^N \sum_{j_3=1}^N \cdots \sum_{j_n=1}^N p_{j_1, \dots, j_n} = \frac{1}{N} & j_2 = 1, \dots, N, \\ \dots, \\ \sum_{j_1=1}^N \sum_{j_2=1}^N \cdots \sum_{j_{n-1}=1}^N p_{j_1, \dots, j_n} = \frac{1}{N} & j_n = 1, \dots, N, \\ 0 \leq p_{j_1, \dots, j_n} \leq 1 & j_i = 1, \dots, N, \\ & i = 1, \dots, n. \end{array} \right.$$

The function $\overline{m}_+(s)$ is the solution of an analogous LP.

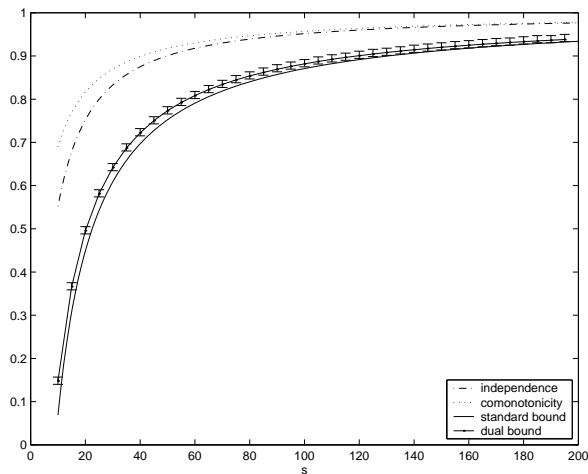


Figure: Range for $\mathbb{P}[X_1 + X_2 + X_3 < s]$ for a Pareto(1.5, 1)-portfolio

Some remarks on this plot

- The ranges for the true solutions have been calculated solving the two LPs with $N = 180$ and using ILOG CPLEX[®] C Callable Libraries (a powerful tool).
- Switching to $n = 5$ drastically lowers the quality of approximation to $N < 50$.
- The worst VaR does not occur under the comonotonicity assumptions, i.e. VaR is not a coherent measure of risk.

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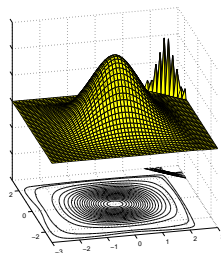
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Non-coherence of VaR

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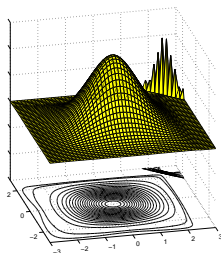
- X_1, X_2 independent but very skew
- X_1, X_2 independent but very heavy-tailed
- $X_1, X_2 \sim N(0, 1)$ but special dependence, see picture below.



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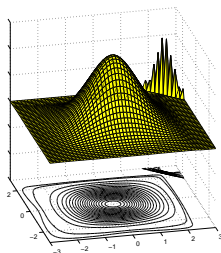
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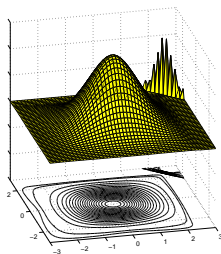
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Bounds on Value-at-Risk

α	$\text{VaR}_\alpha(\sum_{i=1}^3 X_i), \text{exact}$		$\text{VaR}_\alpha(\sum_{i=1}^3 X_i), \text{upper bound}$	
	independence	comonoton.	dual	standard
0.90	7.54	8.85	14.44	15.38
0.95	9.71	12.73	19.50	20.63
0.99	16.06	25.16	35.31	37.03
0.999	29.78	53.99	69.98	73.81

Table: Range for VaR for a Log-Normal(-0.2,1)-portfolio.

Bounds on Value-at-Risk

α	$\text{VaR}_\alpha(\sum_{i=1}^{10} X_i)$		$\text{VaR}_\alpha(\sum_{i=1}^{100} X_i)$		$\text{VaR}_\alpha(\sum_{i=1}^{1000} X_i)$	
	dual	standard	dual	standard	dual	standard
0.90	0.669	1.485	11.039	149.850	150.162	14998.500
0.95	1.353	2.985	22.227	229.850	301.823	29998.500
0.99	2.985	14.985	111.731	1499.850	1515.111	149998.500
0.999	68.382	149.985	1118.652	14999.850	15164.604	1499998.500

Table: Upper bounds for $\text{VaR}_\alpha(\sum_{i=1}^n X_i)$ of three Pareto portfolios of different dimensions. Data in thousands.

Conclusions

The worst-possible VaR for a non-decreasing function of dependent risks can be calculated when:

- some information on the copula of the portfolio is provided
- the portfolio is two-dimensional

When no dependency information is given, the problem gets much more complicated and we provide a new bound which we prove to be better than the standard one generally used in the literature.

Conclusions

The worst-possible VaR for a non-decreasing function of dependent risks can be calculated when:

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Extensions

- Non-homogeneous portfolios (operational risk);
- Other portfolio functions ψ ;
- Multivariate marginals; see Embrechts and Puccetti (2005);
- Other risk measures; see Embrechts et al. (2005).
- For a textbook treatment, see

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